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Developing Algebraic Thinking in the Earlier Grades: A Case Study of the Chinese Elementary School Curriculum[1]

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Abstract: This paper presents a case study of algebraic thinking in the Chinese elementary school mathematics curriculum. The overarching goal of learning algebra in the Chinese elementary curriculum is to help students better represent and understand quantitative relationships. The algebraic emphasis in Chinese elementary school mathematics is related to three of the four goals specified in the NCTM's Principles and Standards for School Mathematics. Equation and equation solving are the big ideas emphasized by the curriculum. Other big ideas are ratio and proportion, variables, and function sense. This curriculum is intended to develop at least three thinking habits for students: (1) Examine quantitative relationships from different perspectives; (2) Relate reverse operations with equation solving, and (3) Generalize from specific examples.

Introduction

Algebra has been characterized as a gateway to opportunities, but too many students have not been given the key. It is widely accepted that to achieve the goal of "algebra for all" in around the world, students in the early grades should have experiences that prepare them for more-sophisticated work of algebra in the middle and high school (NCTM, 2000). However, curriculum developers, educational researchers, teachers, and policy makers are just beginning to think about and explore the kinds of mathematical experiences elementary and middle school students need in order to prepare them for the formal study of algebra in the later grades (Bednarz, Kieran, & Lee, 1996; Carpenter, Franke, & Levi, 2002; Kaput, 1999; Mathematical Sciences Education Board, 1998; Moyer, Driscoll, & Zawojewski, 1998; NCTM, 2000; Schifter, 1999). For example, one of the challenges teachers in the United States face is the lack of a coherent K-8 curriculum that can provide students with rich informal and formal experiences in

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algebra (Schmidt et al., 1996). Knowledge of the curriculum and instructional practices in one country may increase teachers' ability to address the issues and challenges that hinder their students' learning of mathematics in another country.

The purpose of this paper is to examine how the Chinese elementary mathematics curriculum was designed to develop algebraic thinking for students in the earlier grades. It should be indicated that the focus of this paper is neither on evaluating the Chinese elementary mathematics curriculum nor on comparing Chinese and U.S. curricula. Instead, the focus is on providing a case study of the Chinese elementary curriculum on developing algebraic thinking. This case study is based on the analysis of three interrelated dimensions of the Chinese elementary school mathematics curriculum: (1) goal specification, (2) content coverage, and (3) process coverage (see Chapter 1 in this special issue).

In the Chinese educational system, there are 6 years in elementary school, 3 years in junior high school, and 3 years in senior high school (6-3-3 system). Recently, some provinces have been experimenting with a 5-year elementary school, 4-year junior high school, and 3-year high school system. This case study is based on the analysis of the National Syllabus for Elementary School Mathematics issued by the Chinese Ministry of Education as well as students' edition of textbooks and teacher's reference books published by People's Education Press designed for elementary schools in the 6-3-3 system.

In this paper, the analysis and results from the case study of the Chinese elementary school mathematics curriculum are reported in four sections. In the first section, the results from the analysis of the goals are presented. The next three sections present results from analysis of three big ideas related to equation and equation solving, ratio and proportion, and variables and function sense. The discussion of the thinking habits the curriculum is designed to foster is embedded in the sections of equation and equation solving, ratio and proportion, and variables and function sense. This paper ends with a summary of the analysis and results from the case study.

Analysis of Goals

The overarching goal of learning algebra in the Chinese elementary curriculum is to better represent and understand quantitative relationships (Chinese Ministry of Education, 1998). The main focus is on equation and equation solving. The development of ideas related to equation and equation solving in the Chinese elementary mathematics curriculum can be divided into three interrelated stages: (1) the intuitive stage, (2) the introduction stage, and (3) the application stage. The curriculum for grades 1 to 4 consists of the intuitive stage. In this stage, variables, equations, equation solving, and function sense are permeated into the arithmetic analysis of quantitative relationships. The first half of grade 5 contains the introduction stage where equations and equation solving are formally introduced. After the formal introduction of equation and equation solving, equation solving is applied to learning about mathematical topics in the last half of grades 5 and grade 6. These mathematical topics include fractions, percents, statistics, and proportional reasoning.

The intuitive stage sets up the foundation for the formal introduction of equation and equation solving. The introduction stage has three sets of objectives for learning equation and equation solving:

(1) Students understand the meaning of using letters to represent numbers and are able to use letters to represent both numbers and quantitative relationships that are familiar to students. Students are able to find the values of an expression with letters when the values of the letters are specified.

(2) Students understand what an equation is and what the solution to an equation is; and

(3) Students are able to solve two- or three-step application problems using equations. Students can flexibly select either an arithmetic approach or an algebraic approach to solve application problems.

In the application stage, the main goal is to deepen students' understanding of quantitative relationships in various situations as well as to enhance their ability to analyze and solve problems embedded in the content areas of fractions, percents, statistics, and proportional reasoning.

Since ratio and proportion deal with quantitative relationships between or among quantities, this topic plays a vital role in developing students' algebraic thinking in Chinese elementary school mathematics. Proportional relationships provide a powerful means for students to develop algebraic thinking and function sense. There are three sets of objectives for learning ratio and proportion in the Chinese elementary school curriculum.

(1) Students understand the meaning and properties of ratio, and its relationship with fractions and divisions. Students are able to solve application problems involving ratio.

(2) Students understand the meaning and properties of proportion. In particular, they should understand direct and inverse proportional relationships and are able to identify when two quantities have direct or inverse proportionality.

(3) Students are able to solve application problems using the knowledge of ratio and proportions.

The concepts of variables and functions are not formally defined in Chinese elementary school mathematics. Therefore, no specific learning objectives are listed in the Chinese curriculum. However, the curriculum is designed to permeate variable and function ideas throughout the curriculum to develop students' function sense. According to the teacher's reference book, teachers should take every opportunity to develop students' variable and function ideas, starting in the first grade. Students should understand how two quantities are related and how the relationships between two quantities can be represented in various forms. The teacher's reference book recommends that the development of students' knowledge of variables and function ideas must play a vital role in students' learning of mathematics.

Analysis of Big Ideas: Equation and Equation Solving

Intuitive stage of equation and equation solving

The idea of equation and equation solving starts at the very beginning of first grade Chinese elementary textbooks (Division of Elementary Mathematics, 1999a, 1999m). In Chinese elementary schools, addition and subtraction are introduced simultaneously, and subtraction is introduced as the reverse operation of addition (Division of Elementary Mathematics, 1999a). The idea of equation and equation solving is permeated throughout this introduction of subtraction. Students are guided to think about the following question: Since 1 + 2 = 3, □ + 1 = 3? In other words, what number should we place in the box so that the number + 1 = 3? Throughout the first grade, students are consistently asked to solve similar problems. For example, they are asked to find the number in ( ) so that 9 + ( ) = 16 or 50 + ( ) = 59.

In the second grade, multiplication and division with whole numbers are introduced in Chinese elementary school mathematics (Division of Elementary Mathematics, 1999c, 1999d). Division is first introduced using equal sharing or dividing. Division is also introduced as a reverse operation of multiplication: what multiplies by 2 = 8? That is, if □ × 2 = 8, what is the number in □? The second grade textbooks also include mixed operation problems involving unknowns, such as 7 × 6 - □ = 18 and 5 × □ + 8 = 18. There are similar problems in the third and fourth grade textbooks involving numbers and operations with multi-digits whole numbers, fractions, and decimals (Division of Elementary Mathematics, 1999e, 1999g).

Introduction stage of equation and equation solving

In the first half of grade 5, the Chinese elementary textbook includes a unit on Simple Equation (Division of Elementary Mathematics, 1999i, 1999u). This particular unit consists of three sections: (1) using letters representing numbers and quantitative relationships, (2) solving simple equations, and (3) using equation solving to solve application problems.

Using letters representing numbers and quantitative relationships. This is not the first time that letters are used to represent numbers. Immediately before the Simple Equations unit is the unit for Finding Areas of Polygons. In this unit, letters are introduced to represent the formulas of finding areas of squares, triangles, rectangles, and trapezoids. And also, before the Simple Equations unit, students in the second a half grade 4 have the opportunity to use letters to represent properties of operations such as the commutative law for addition (a + b = b + a), the associate law for addition [(a + b) + c = a + (b + c)], and the distributive law of multiplication over addition [a(b + c) = ab + ac]. The teacher's reference book clearly points out that the purpose of using letters to represent the formulas for finding the area of a polygon is to show the generality of the ways to find areas of polygons (Division of Elementary Mathematics, 1999u). For example, although there are a variety of rectangles, the area of any kind of rectangles can be found by multiplying width with the length. This generality emphasis is consistent with the findings about students' mathematical thinking (Cai & Huang, 2002).

After a review of using letters to represent formulas and numbers in computational properties, using letters to represent numbers are introduced. There are two distinct features of using letters to represent numbers in this unit in contrast to the discussion of using letters to represent formulas of finding areas and computational properties. The first distinct feature is the focus on using letters to represent quantitative relationships, instead of just using letters to represent a quantity. For example, if P represents the perimeter of a square and s represents the length of a side, then C = 4s. The teacher's reference book states that C = 4s represents how the length of the side of a square related to its perimeter. The second distinct feature is that the values letters represent varies. The teacher's reference book suggests that teachers should explicitly guide students to explore how the values of one quantity will be changed as a result of the changes in another quantity (Division of Elementary Mathematics, 1999u). In the above example, students are guided to explore what happens when the length of a side in a square changes, and how the perimeter of the square will change accordingly. In the student text, several familiar quantitative relationships are included, such as distance = speed × time (d =vt) and profit = total revenue – total cost (p = r – c).

Solving simple equations. In this section, sequential lessons are included in the student textbook. Students are first introduced how to solve an equation like x + a = b. The Chinese textbook uses the balance model to introduce the concept of an equation and shows how to solve an equation like x + a = b (Division of Elementary Mathematics, 1999i). An equation is described as an equality involving unknowns. The teacher's reference book recommends that teachers should help students distinguish between the equation's solution and the process of solving an equation

(Division of Elementary Mathematics, 1999u). In this part of the textbook, the emphasis is on the students' understanding of what an equation is, what the solution of an equation is, and how to use the balance model to solve an equation.

Then students are taught how to use the additive property to solve an equation like $x \pm a = b$. If we add or subtract a on both sides of $x \pm a = b$, the equality still holds. Using examples, students are explicitly taught that the goal of solving an equation is to find the unknown. In order to find the unknown, students usually use the additive property to equivalently transform one form of equation into another. After students are taught equation solving of $x \pm a = b$ by using additive properties, they are introduced equation solving of $ax \pm b = c$ by using both additive and multiplicative properties. Moreover, at least one lesson is recommended to learn how to solve an equation like $ax \pm bx = c$. This arrangement is designed to help students put all the lessons together.

Solving application problems. The teacher's reference book recommends 6 lessons for the section on using letters to represent numbers and 5 lessons for the section on solving simple equations (Division of Elementary Mathematics, 1999u). However, it recommends 12 lessons for the section on solving application problems. The recommended time for the section on solving application problems highlights the emphasis the Chinese elementary mathematics curriculum places on this section. Like the section of solving simple equations, this section is designed sequentially according to the difficulty level of the application problems. However, regardless of the difficulty level of application problems, the focus is on helping students understand quantitative relationships.

In this section, students are first introduced to one- or two-step application problems, involving "how many or how much more" quantitative relationships (additive). The following problem is typical of the problems introduced early in this section: *Xiao Qing purchased two batteries. She gave the cashier 6 yuans and got 0.4 yuan in change back. How much does each battery cost?* In this example, the quantitative relationship involves the amount of money paid to the cashier, the change, and the cost of the two batteries. The teacher's reference book recommends that teachers allow students to represent the quantitative relationship in different ways, such as

The amount of money paid to the cashier – the cost of the two batteries = the change
The cost of the two batteries + the change = the amount of money paid to the cashier
The amount of money paid to the cashier – the change = the cost of the two batteries

Students are guided to compare these three quantitative relationships and see how each of the quantitative relationships can be represented using an equation. According to the teacher's reference book, representing quantitative relationships in different ways will not only help students develop deeper understanding mathematics; but also will help them develop their flexibility of using equations to solve application problems (Division of Elementary Mathematics, 1999s, 1999t, 1999u, 1999v). After the one- or two-step application problems, students are taught two- and three-step application problems, which involve "how many times more or less" quantitative relationships (multiplicative). The following problem is an example of this type of problem: *Jia-Jia Elementary School has 84 boys. The number of boys is three times and 15 more than the number of girls. How many girls are there in the school?*

At the end of this section, the teacher's reference book recommends three lessons to compare the arithmetic and algebraic approaches to solve application problems. For example, teachers talk about four ways to solve the following problem: *Liming Elementary school has funds to buy 12 basketballs at 24 yuan each. Before buying the basketballs, they decided to spend 144 yuan of the fund for some soccer balls. How many basketballs can they buy*?

Solution 1: $(24 \times 12 - 144) \div 24 = 144 \div 24 = 6$ basketballs.
Solution 2: $12 - 144 \div 24 = 6$ basketballs.
Solution 3: Assume that the school can still buy x basketballs. $24x + 144 = 24 \times 12$.
Therefore, $x = 6$ basketballs.
Solution 4: Assume that the school can still buy x basketballs. $24x = 24 \times 12 - 144$.
Therefore, $x = 6$ basketballs.

Chinese textbooks contain a number of examples showing how to solve problems using both arithmetic and algebraic strategies. There are three objectives in teaching students to solve problems both arithmetically and algebraically: (1) to help students attain an in-depth understanding of quantitative relationships by representing them both arithmetically and algebraically; (2) to guide students to discover the similarities and differences between arithmetic and algebraic approaches, so they can make a smooth transition from arithmetic to algebraic thinking, and (3) to develop students' thinking skills and flexibility in using appropriate approaches to solve problems.

In addition to that, solving an application problem using both an arithmetic and an algebraic approach helps students make a smooth transition from arithmetic to algebraic thinking. Since all of the problems discussed in this section can be solved arithmetically, it is common for students not to see why they need to learn equation-solving approach to solve problems. However, by using both approaches, students can see the advantages of using equations to solve an application problem. For example, setting up an equation has advantages when solving the following problem: *Yining Elementary School has 84 boys. The number of boys is three times, and 15 more than, the number of girls. How many girls are there in the school*?

Application stage of equation and equation solving

After the students are formally introduced to equations and equation solving, they are provided opportunities to use an equation-solving approach to solve application problems when they learn statistics, percents, fractions, and ratios and proportions (Division of Elementary Mathematics, 1999j, 1999k, 1999l). Again, the main purpose of designing the curriculum in this way is to deepen the students' understanding of quantitative relationships and to help students appreciate the equation-solving approach. This design reinforces the smooth transition from arithmetic to algebraic thinking. Several of these examples are included below:

Example 1 (Statistics): *There are five households and the number of people in the first four households is 6, 4, 3, and 4, respectively. We know the mean number of people in the five households is 4. How many people are there in the fifth household?*

Solution 1: Let x be the number of people in the fifth household. $6 + 4 + 3 + 4 + x = 5 \times 4$. $x = 3$ by solving the equation for x. Therefore, the fifth household has three people.

Solution 2: $6 + 4 + 3 + 4 = 17$. $5 \times 4 = 20$. $20 - 17 = 3$. Therefore, the fifth household has three people.

Example 2 (Percent): *A factory modified its production procedures. After that, the cost for making one product is \$37.4 which is 15% lower than the cost before the modification of production procedures. What was the cost for making the same product before the modification of production procedures?*

Solution 1: If the cost for making a product before the modification can be viewed as Unit 1, then the current cost is 15% less than the cost before modification.

Let x = the cost before the modification.

$x - 15\%x = 37.4$

$(1 - 15\%)x = 37.4$

$85\%x = 37.4$

$x = 37.4 \div 85\%$

$x = 37.4 \div 0.85$

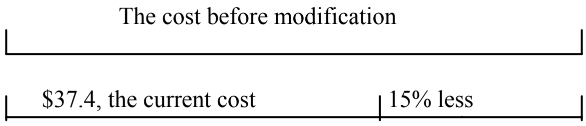
$x = 44$

Therefore, the cost before the modification is \$44.

Solution 2: The cost for making a product before the modification can be viewed as Unit 1, so \$37.40 is 15% less, or 1 - 15% = 85% of the cost before the modification. Therefore, the cost before the modification is $37.40 \div 85\% = 44$.

Answer: The cost before the modification is \$44.

Solution 3: Similarly, the cost for making a product before the modification can be viewed as Unit 1, shown in the figure below. Since the current cost is 15% less than the cost before



the modification, so the current cost is 85% of the previous cost. Therefore, the cost before the modification is 37.40 ÷ 85% = 44.
Answer: The cost before the modification is $44.

Solution 4: Because the percent also represents a ratio, the cost after the modification to the cost before the modification is 85 to 100. It will be the same as $37.4 to *x*, which is the cost before the modification.
Hence, 37.4/x = 85/100, x = 44.
Therefore, the cost before the modification is $44.

Example 3 (ratio and proportion): *The Johnson School District brought some math books. The district distributed these books to the Edison Middle School and Radnor Middle School using a 4 to 5 ratio. Edison Middle School received 200 books. How many books are there in total?*

Solution 1: Since these books were distributed according to the 4 to 5 ratio, we can imagine that they were divided into nine equal parts. Edison Middle School got 4 out of the nine parts, which is 200 books. Therefore, each part represents 50 books, resulting from 200 ÷ 4 = 50. 50 × 9 = 450. Therefore, there are 450 books in total.

Solution 2: Let x be the total number of books the district has. Since these books were distributed according to the 4 to 5 ratio, we can image that they were divided into nine equal parts. Edison Middle School got 4 out of the 9 parts; therefore, the ratio of the number of books Edison Middle School got to the total number of books should be 4:(4 + 5). Thus, we have 200/x = 4/(4+5). By solving the equation for x, the total number of books the district had is 450.

Solution 3: Let x be the number of books Radnor Middle School got. Since the district distributed these books to Edison Middle School and Radnor Middle School according to 4 to 5 ratio, the ratio of the number of books Edison Middle School got to the number of books Radnor Middle School got should be 4:5. Therefore, 200/x = 4/5. By solving the equation for x, the total number of books Radnor Middle School got is 250. Therefore, the total number of the books should be 200 + 250 = 450 books.

Solution 4: The total number of books the school district has can be considered as a "whole." Since the district distributed these books to Edison Middle School and Radnor Middle School according to the 4 to 5 ratio, Edison Middle School got 4/9 of the total books. Let x be the total number of books the district has. x × 4/9= 200. By solving the equation for x, the total number of books the district had is 450.

The teacher's reference book strongly recommends that teachers should allow students to come up with their own solutions and compare the various solutions students come up with (Division of Elementary Mathematics, 1999v, 1999w, 1999x). Using these various approaches and representations, students can gain insight into the quantitative relationships in various contents and contexts and develop their ability to model problem situations using equations. The arithmetic approach holds the premise for developing students' ability to reason qualitatively (Post, Behr, & Lesh, 1988). The equation approach can be useful and efficient tool for problem solving. Through solving the problems in different ways, students can make connections between various methods. Such comparison is beneficial for both reinforcing what students had learned and setting up stages for learning new ways to solve a problem.

Analysis of Big Ideas: Ratio and Proportion

In Chinese elementary school mathematics, another big idea in algebraic thinking is Ratio and Proportion. The topics related to ratio and proportion are divided into three units: ratio, proportion, and application of ratio and proportion . The ratio unit is included in the first half of the sixth-grade curriculum (Division of Elementary Mathematics, 1999k). The next two units are included in the second half of the sixth-grade curriculum (Division of Elementary Mathematics, 1999l). The teacher's reference book recommends five lessons (about 40-45 minutes each) for teaching the concept of ratio, eight lessons for teaching the concept of proportion, and four lessons for teaching the applications of ratio and proportion [2] (Division of Elementary Mathematics, 1999w, 1999x).

Introducing the concept of ratio

In the Chinese elementary mathematics curriculum, ratio is introduced in the unit on division with fractions in the first half of the sixth-grade textbook. The concept of ratio is clearly defined as the comparison of two quantities with multiplicative relationships. Interestingly, ratio in Chinese means "to compare or comparison." The teacher's reference book highlights both the importance of ratio and the difficulties students have in order in understanding the multiplicative nature of the concept of ratio. Ratio is introduced and treated both as a concept and as an operation. When it is introduced, rather than just telling students what a ratio is and how it is represented, division is used as a bridge to connect the concept of ratio and its representations. Following is an example illustrating how Chinese teachers usually introduce the concept of ratio: *Miller Middle School has 16 sixth-grade*

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[2] Compared to an earlier edition of the Chinese elementary textbook, the number of recommended lessons for the applications of ratio and proportion was reduced from 8 lessons to 4 lessons. See Cai & Sun (2002) for information about ratio and proportion in an earlier edition of Chinese elementary school mathematics.

students and 12 of them said that they are basketball fans. The remaining students are not basketball fans. How could we describe the relationship between the students who are basketball fans and those who are not?

Because there are 16 students and 12 of them are basketball fans, the number of students who are not basketball fans is 16 – 12 = 4. By comparing these two numbers, some students may say that there are 8 more students who are basketball fans than those who are not basketball fans (12 – 4 = 8). Other students may say: There are 3 times as many basketball fans as non-basketball fans (12 ÷ 4 = 3). Yet others may say: among these 16 students, for every 3 students who like basketball, there is one student who does not like basketball. Students in the first case used additive reasoning to describe the relationship between the students who are basketball fans and who are not. Students in the latter cases used multiplicative reasoning to describe the relationship between the students who are basketball fans and the students who are not. The multiplicative relationship between two quantities is defined as a ratio in Chinese textbooks. There are two ways to describe how two quantities are related: additive versus multiplicative. The ratio of the two quantities involves the comparison of the quantities using the multiplicative relationships. The Chinese teacher's reference book indicates that such multiplicative relationships are considered to be important, but difficult for students to learn (Division of Elementary Mathematics, 1999w).

In the above example, the Chinese mathematics teachers emphasize that when using a ratio to describe the relationship between the two groups of students, students need to begin with the knowledge they have learned (division 12 ÷ 4) and use this knowledge to describe the new situation (ratio of 3:1). Because students have studied division of integers and understand that a fraction also represents division, the connections among ratios, division, and fractions are easily made. The division a÷b is the operation of the ratio of a:b, and the fraction a/b is the result of the ratio operation. Chinese teachers call a/b the Value of the Ratio of a:b. By using the term "the value of the ratio," which is represented by the fraction a/b, students are able to make connections among a:b, a÷b, and a/b. The figure below shows the way that Chinese mathematics teachers describe the relationship between the two groups of students.

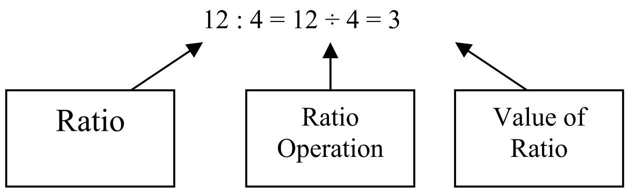


Figure 1. Ratio, Ratio Operation, and Value of Ratio

Chinese teachers explain that the ratio of the students who like basketball to the students who do not like basketball is 12 : 4 and the value of this ratio is 3. Also, a property on the value of ratio is introduced in the Chinese curriculum: when the two quantities in a ratio are multiplied by the same none-zero number, the value of the ratio remains the same. The property is introduced by relating it to reducing fractions. This property is very helpful for students when making comparisons between different ratios and for studying the concept of proportion ((Division of Elementary Mathematics, 1999k).

Introducing the concept of proportion

The concept of proportion, which is introduced in the second half of the sixth-grade textbook, is built upon the understanding of ratio. Chinese mathematics teachers have given a great deal of attention to smoothly connecting them. Scale drawing is one of the means of bridging the connection between ratio and proportion. The teachers' reference books indicated that if students have a solid understanding of scale drawing, they can quickly grasp the sense of proportionality. Following is an example of how scale drawing helps students develop a sense of proportion: *A corporation is going to build an office building. Two centimeters on the blueprint represents 5 meters on the ground. If the building is 18 centimeters long on the blueprint, what is the actual length of the building on the ground?*

To solve this problem, students need to find the ratio of the distance on the blueprint to that on the ground. Chinese teachers emphasize the importance of using the same measurement unit for the two quantities to find the ratio. Therefore, five meters should be converted into 500 centimeters. In the context of scale drawing, ratios are usually reduced to unit ratios.

2 cm : 5 m = 2 cm : 500 cm
= 2 : 500
= 1 : 250 (or 1/250)

Therefore, the ratio of the distance on the blueprint to the distance on the ground is 1:250. Initially, this sounds like a ratio problem, but while solving this problem, students are developing an understanding of proportion.

In this case, the initial ratio is a ratio of two concrete quantities. By simplifying the ratio, it becomes an abstraction. This process, moving from concrete to abstraction reflects the relationship between these two ratios. When teaching scale drawing, the Chinese mathematics curriculum emphasizes the meaning of scale and the role it plays in terms of helping students determine proportion. With a good understanding of scale drawing, students should be able to make a smooth transition from ratio to proportion.

Proportion, in the Chinese mathematics curriculum, is described as two ratios that have the same value (Division of Elementary Mathematics, 1999l). The following example demonstrates how a Chinese teacher would introduce the concept of proportion: *Car A traveled 120 miles in 2 hours. Car B traveled 180 miles in three hours. Describe which car traveled faster.*

For Car A, the ratio of distance to time is 120:2 and for Car B, the ratio of distance to time is 180:3. They have the same Value of Ratio, 60 miles per hour, which is the speed that each car traveled. That is, 120/2 = 180/3 = 60. Thus, the concept of proportion can be naturally developed: two ratios that are equal.

In order to help students develop proportional reasoning, they need to understand the important role that direct proportionality and inverse proportionality play. Both direct proportionality and inverse proportionality represent a relationship between two quantities, and the change of one quantity directly impacts the change of another quantity. In Chinese schools, students are provided with various examples to distinguish direct proportionality from inverse proportionality between quantities. They are also asked to make up problems in which quantities have direct proportionality and inverse proportionality (Division of Elementary Mathematics, 199l).

It is very important that students are provided with concrete examples so that they can develop an understanding of direct proportionality. However, teachers' reference books pointed out that teaching proportionality cannot stop at the concrete level. A mathematical representation of direct proportionality needs to be generalized. Using the above problem, once students see how time and distance are related, it is natural to generalize the relationships among speed, distance, and time, which is $s = d/t$. From this mathematical representation (formula), students can easily see that when s is a constant, d and t are directly proportional. The Chinese textbook introduced the direct proportional relationship symbolically as $y/x = k$, where x and y represent two related quantities and k represents the constant. In addition to that, the Chinese teacher's reference book includes the properties of proportionality as an appendix.

Inverse proportionality is introduced in the Chinese elementary mathematics curriculum right after direct proportionality is taught. This gives students a good opportunity to understand proportionality from a broader point of view. Meanwhile, students can also see how direct proportionality and inverse proportionality are related and distinguished from each other. They both represent a relationship between two quantities, and the change of one quantity impacts the change of another quantity. Like direct proportionality, the Chinese textbook introduced the inverse proportional relationship symbolically as xy = k, where x and y represent two related quantities and k represents the constant. Here is an example from a Chinese mathematics textbook: *A person plans to travel by bike to another village that is 60 km away. Discuss how the speed of the bike and time needed to travel are related.*

Student can create a table to represent this relationship (See Table 1 below). When students examine the table, they can see that as the speed increases, the time needed to travel 60 km decreases. Furthermore, the product of the corresponding quantities (speed and the time) is a constant that is the actual distance between the two villages. This relationship represents inverse proportionality. When students understand the relationship between speed and time, they can generalize it to a mathematical expression: $d = s \times t$. This generalization, again, is very important so students can understand how inverse proportionality is represented mathematically.

Table 1.
An Example of Inverse Proportionality

Speed (km/hour)	1	2	3	4	5	6	……
Time	60	30	20	15	12	10	……

It is also important to teach direct proportionality and inverse proportionality by helping students determine when two quantities are directly or inversely proportional. When the value of the ratio of the two quantities is a constant, then the two quantities are directly proportional. When the product of the two quantities is a constant, then the two quantities are inversely proportional. With a solid understanding of ratio, proportion, direct proportionality, and inverse proportionality, students have a strong foundation to develop their ability in reasoning proportionally.

The development of proportional reasoning through problem solving

The Chinese curriculum contains a special unit on using proportional reasoning to solve problems in various contexts. The major learning objective in this unit is to enhance students' understanding of the concepts of ratio and proportion by applying a proportional relationship to solve various problems (Division of Elementary Mathematics, 1999x). Based on the earlier lessons about the concepts of ratio and proportions, students are guided to make connections between direct proportionality and inverse proportionality, connections between what they have learned and what they are learning, and connections between different solution strategies. The Chinese curriculum includes problems with various levels of difficulty, and the teacher's reference book contains discussions of teaching each type of the problems.

In solving these problems, students first need to decide which proportional relationships among the quantities are involved in the problems, and they are encouraged to use different knowledge or approaches. Let's take a look at the following two examples.

Example 1: *John traveled 180 miles in three hours. It took him five hours to go from Chinatown to Germantown at the same speed. What is the distance from Chinatown to Germantown?*
Example 2: *It took John five hours to go from Chinatown to Germantown when he traveled 60 miles per hour. If John wanted to spend four hours going from Chinatown to Germantown, how fast should he travel per hour?"*

This type of problem involves quantitative relationships among distance, time, and speed. Before solving them, students are asked to determine the quantity that is invariant and the two quantities that are covariant. In example 1, the speed is invariant, and there is a direct proportional relationship between the time and distance. In example 2, the distance from Chinatown to Germantown is a hidden known, which is the invariant. With the invariant distance from Chinatown to Germantown, there is an inverse proportional relationship between speed and time. That is, the faster John travels, the less time he takes. Comparing the processes of solving these two problems may enhance students' understanding of the direct and inverse proportional relationships. Through qualitative reasoning and emphasis on invariant quantity, teachers can help students make sense of the standard proportional algorithm.

As indicated before, the Chinese teacher's reference book clearly indicates that teachers should guide students to solve these types of problems using different methods. Students are guided first to use an arithmetic approach, with which they are very familiar, to solve the problem and then to solve it through setting up an equation based on the proportional relationship.

Arithmetic Approach for Example 1: Since John traveled 180 miles in three hours, he traveled 60 miles per hour. It took him 5 hours from Chinatown to Germantown, so the distance from Chinatown to Germantown is $5 \times 60 = 300$ miles.

Setting up an equation based on proportional relationship for Example 1: Let x = the distance from Chinatown to Germantown. Since John travels at the same speed, 180/3 = the speed = x/5. Therefore, we have $180/3 = x/5$. Solving the equation for x yields $x = 300$ miles.

Similarly, students can be asked to solve example 2 in two different ways: (1) $5 \times 60 = 300$ miles. $300 \div 4 = 75$ miles per hour. (2) Let x be the number of miles John traveled per hour when he wanted to complete the journey in four hours. 4x = the distance from Chinatown to Germantown = 5 × 60. Therefore, 4x = 5 × 60. x = 75 through solving the equation for x.

Analysis of Big Ideas: Variables and Function Sense

Variable ideas

The term "variable" is not defined in Chinese elementary school mathematics. However, the teacher's reference book includes the message that variables can represent many numbers simultaneously, they have no place value, and they can be selected arbitrarily (Division of Elementary Mathematics, 1999m, 1999n, 1999o, 1999p, 1999t). In Chinese elementary school mathematics, variable ideas are used in three different ways.

First, variables are used as place holders of unknowns in equation solving. In grades 1-3, a question mark, a picture, a word, a blanket, or a box is used to represent the unknowns in equations (e.g., Division of Elementary Mathematics, 1999c, 1999d, 1999e, 1999f). In the first half of the fourth grade (Division of Elementary Mathematics, 1999g), the letter "x" is first used to represent unknowns in the following problem: *Ding-Ding Elementary School purchased some boxes of chalk. They used 28 boxes, and there are 42 remaining. How many boxes of chalk did the school buy?* At this grade level, students are taught to use the following method to solve this problem: 28 + 42 = 70. This means that the number of boxes of chalk used + the number of boxes left = the total number of boxes purchased. However, the textbook indicates that this relationship can be viewed differently: the total number of boxes purchased – the number of boxes used = the number of boxes left. In this problem, the total number of boxes purchased is unknown, so let x represent the total number of boxes purchased. According to the problem, we have x – 28 = 42. At this grade level, students are not taught to "solve" this equation to find x. Instead, students are guided to use subtraction as an inverse operation of addition to find x. What minus 28 = 42? Since 28 + 42 = 70, so x = 70. The teacher's reference book strongly recommends teachers make explicit comparisons between the two approaches to solve this problem. In the second approach (in grade 5, it is called the equation-solving approach or algebraic approach), the unknown, or the total number of boxes purchased, is represented as x and is directly involved in the solution process. In contrast, in the first approach (in grade 5, it is explicitly called arithmetic approach), the unknown is not used until the total number of boxes purchased is determined at the end. As mentioned before, these types of comparisons of different approaches are frequently found in the textbooks for grades 4, 5, and 6 in China.

In addition, in grades 4, 5 and 6, "x" as well as other letters are introduced as placeholders of unknowns in equation solving. For example, students are asked to solve the following equations: x + 6 = 21 and n/5 = 3/10. In some cases, variables are used as placeholders of unknowns in unfamiliar situations. For example, students are asked to find the numbers in "?" so that $\frac{1}{?}+\frac{1}{?}+\frac{1}{?}=\frac{11}{18}$. Another example is to ask students find the value of O and □ represent if O × □ = 36 and O ÷ □ = 4. Variables are also used as place holders of unknowns when students are asked to translate a mathematical statement into an expression.

Second, variables are viewed as pattern generalizers or as representative of a range of values. For example, variable are used to represent formulas. In the third grade, after examining several specific examples, areas of squares and rectangles are introduced and the formulas are represented using words:

Area of a rectangle = its length × its width
Area of a square = its side × its side

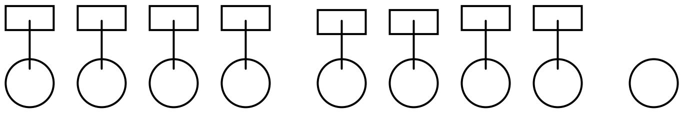
In this grade level, letters are not used to represent formulas in order to help students understand the meanings of these formulas. However, teachers are recommended to emphasize the generalizable nature of the formulas. That is, for any rectangle, its area can be found by multiplying its length and width. In grades 5 and 6, letters are used to represent formulas for finding areas of squares, triangles, rectangles, trapezoids, and circles. Another use of variables as pattern generalizers in Chinese elementary school mathematics is in operational properties. In grades 4 and 5, through examining various examples, such as 3 + 5 = 5 + 3, teachers guide students to represent the pattern using letters, such as a + b = b + a.

Third, variables are used to represent relationships, such as direct proportionality (x/y = k) and inverse proportionality (xy = k). This way of using variables is discussed more thoroughly in the next section.

Function Sense

In the Chinese elementary curriculum, the function concept is not formally introduced. However, the curriculum is designed to permeate function ideas throughout the curriculum to develop students' function sense. The teacher's reference book clearly points out that function ideas deal with the interdependent relationships between quantities. Permeating function ideas into various content areas not only fosters students' learning of the content topics but it also provides a solid foundation for their learning of advanced mathematical topics in middle and high schools. However, the teacher's reference book also points out that students should be provided with as many opportunities as possible to develop function sense at the concrete and intuitive level at the beginning. Then the teachers can gradually guide the students to the relatively abstract level. In elementary school, teachers should be careful not to push too hard to have students come up with generalized conclusions. The curriculum designers and teachers believe that there are many great opportunities to develop students' function sense.

The curriculum is designed to develop students' function sense, starting in the first grade. The function sense is first introduced in the context of comparing and operating with whole numbers at grade 1. In the first half of the first grade textbook, there are a number of examples of comparing two numbers using one-to-one-correspondence. For example, in order to compare 8 with 9, 8 rectangles and 9 circles are drawn (see picture below). One rectangle corresponds to one circle, but there is one circle that does not correspond to any rectangles. Therefore, 9 is bigger than 8.



Addition with whole numbers is also used to develop students' function sense. Students are asked to examine a series of additions (see example below) and then to point out their observations. The purpose of using an example like this is to let students understand the changes of the sum when one of the addends changes. There are also opportunities like this later on when students are learning four operations with decimals, fractions, and percents as well as learning multiples and factors.

$8 + 1 = 9$
$8 + 2 = 10$
$8 + 3 = 11$
$8 + 4 = 12$
$8 + 5 = 13$
$8 + 6 = 14$
$8 + 7 = 15$

In the third grade, unit rate is introduced using familiar situations, such as speed. For a given constant speed, students are asked to discuss how the distance traveled changes when the time changes. Students are given tables to see the relationship between the time traveled and the distance traveled with a given speed. At this grade level, the teacher's reference book recommends only using words (instead of letters) to represent functional relationship between distance and time with given speed.

In grade 5, the unit "Simple Equations" includes many opportunities to develop students' function sense. For example, students are asked to use letters to represent the following quantitative relationships: a sister is 4 years older than her brother. In this particular quantitative relationship, students don't know the ages of either the sister or the brother, but the sister is 4 years older than her brother. The textbook includes an expression (sister's age = a + 4) and a table to illustrate the one-to-one correspondence between the brother's age and sister's age. The teacher's reference book recommends that teachers explicitly point out when the sister's age is determined, then the brother's age is determined and vice versa. The most challenging problem in this unit is the following: *a and b are two natural numbers and a + b = 100. What is the maximum and minimum value of the product of a and b?*

In grade 6, multiple representations are used to represent functional relationships between two quantities. These functional relationships are embedded in learning mathematical topics about circles, statistics, and proportional reasoning. For the unit on circles, the textbook not only discusses how the circumference and area of a circle are related to its radius, but it also investigates how the circumference and area of a circle are related. Students are asked to complete the following table in order to see various relationships with circles. From this table, students can see how the changes in the length of a radius, the circumference and area of the circle are changed as well as the ratio of area to circumference.

Table 2.
Quantitative Relationships with a Circle

Radius (R)	1 cm	2 cm	3 cm	4 cm	5 cm
$C = 2\pi R$					
$A = \pi R^2$					
$A/C = R/2$					

In the unit on proportionality in grade 6 of Chinese elementary school mathematics, the textbook represents the relationship using tables, symbolic expressions and graphs. In particular, students are first asked to observe the following two tables to determine how the distance and time are related in Table 3a and how speed and time in are related in Table 3b. As was indicated before, for a given speed, distance and time have a direct proportional relationship. When distance is given, speed and time have an inverse proportional relationship. The textbook also notes that graphs can be used to represent direct and inverse proportional relationships, as shown in Figure 2.

Table 3a.
Relationship between Distance and Time

Distance (KM)	5	10	25	50	100
Time (hours)	1	2	5	10	20

Table 3b.
Relationship between Speed and Time

Speed (KM/H)	100	50	20	10	5
Time (hours)	1	2	5	10	20

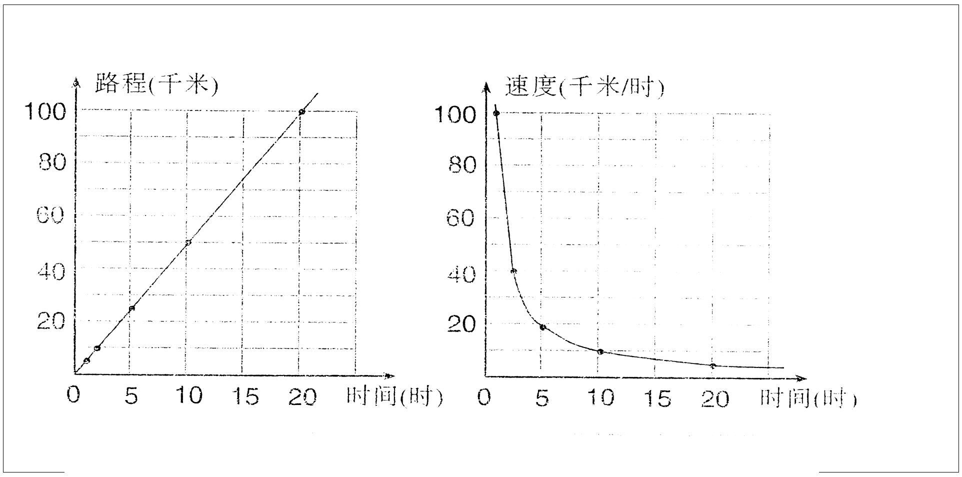


Figure 2. Graphs show direct and inverse proportionalities

The last unit in the Chinese elementary school mathematics curriculum is basic statistics. In this unit, tables and line graphs are used extensively to show the relationships between two variables. For example, students are given average temperatures of each month in 1997. According to the data in the table, students are guided to plot each point on graph paper and then to connect the plotted points for a line graph (see Figure 3). Based on the graph, students are asked to analyze changes by answering questions, such as in which months, did the average temperature increase the fastest or slowest? It should be indicated that the analysis of changes are qualitative and used in a global sense. Detailed, quantitative analysis of changes is not involved.

Table 4.
Average Temperatures of Each Month in 1997

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sep	Oct	Nov	Dec
Average Temp (°C)	2	5	10	16.5	22	28	32	32.5	26	19	11.5	5

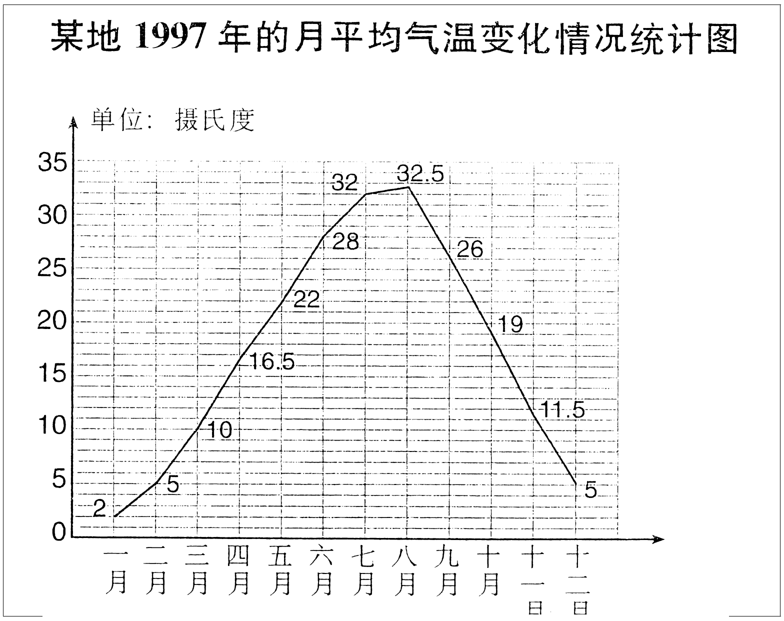


Figure 3. Graph showing average temperature in 1997

Conclusion

This paper presented a case study of algebraic thinking in Chinese elementary school mathematics curriculum. The overarching goal of learning algebra curriculum is to help students better represent and understand quantitative relationships. The main focus is on equation and equation solving. Variables and function sense are permeated into the arithmetic analysis of quantitative relationships in the curriculum for grades 1 to 4. Equations and equation solving are formally introduced in the first half of grade 5. After the formal introduction of equation and equation solving, equation solving is applied to learn mathematical topics, such as fractions, percents, statistics, and proportional reasoning in the last half of grades 5 and grade 6. In the Principles and Standards for School Mathematics, NCTM listed four goals related to the algebra strand: (1) Understand patterns, relations, and functions; (2) Represent and analyze mathematical situations and structures using algebraic symbols; (3) Use mathematical models to represent and understand quantitative relationships; and (4) Analyze change in various contexts. The algebraic emphasis in Chinese elementary school mathematics is related to the first three goals. For the fourth goal, only qualitative analysis of changes is included. The fourth goal is addressed fully when the concept of function is formally introduced in Chinese junior high schools.

The Chinese elementary school curriculum is intended to develop at least three thinking habits for students. The first thinking habit is to examine quantitative relationships from different perspectives. Students are consistently encouraged and provided with opportunities to represent a quantitative relationship both arithmetically and algebraically. Furthermore, students are asked to make comparisons between arithmetical and algebraic ways of representing a quantitative relationship. Throughout the Chinese elementary school curriculum, there are numerous examples and problems in which students need to identify quantitative relationships and represent them in multiple ways.

The second thinking habit is to relate reverse operations with equation solving. Starting in the first grade, subtraction is defined as the reverse of addition. Although students are not told the term, they learn to solve equations starting at grade 1 and continue throughout the entire curriculum.

The third thinking habit is to generalize from specific examples. By examining specific examples, students are guided to create generalized expressions. This habit of mind is particularly instilled when formulas for finding perimeters, areas, and volumes are introduced, when operational laws are presented, or when the averaging algorithm is discussed.

This paper provided information from a case study of Chinese elementary school mathematics curriculum on algebraic thinking. Because of the complexity and culturally-bounded nature of curriculum development and teaching, we must cautious about transposing curriculum and teaching practices from one country to the another. However, the information from a case study like this should be an important reference for educators and curriculum developers in different countries when they try to establish a mechanism for providing students experiences with algebraic ideas and thinking in the earlier grades.

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